On Cobweb posets tiling problem

The method of the cobweb tiling

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1. Introduction and notation

The proof on existence the partition of some cobweb-admissible sequences F, that introduced in before note gives as a method to create exactly one partition of certain layer. Naturally, there are more different partitions of certain layer, but we know surely, that exists at least one. In this note we'll present rules to obtain a partition of any layer $\langle \Phi_k \to \Phi_n \rangle$. All of definitions are follows with sources [1, 2].

Definition 1

As a *block* σP_m we mean sequence of vertices' sets which it blocks cover i.e.

$$\sigma P_m[\varphi_1, \varphi_2, ..., \varphi_m] \equiv (\varphi_1, \varphi_2, ..., \varphi_m)$$

where $\varphi_s = \{ f_i \ : f_i \in \{1,2,...,(k+\sigma \cdot s)_F\} \wedge i = 1,2,...,\sigma \cdot s \}$ is a set of covered vertices on s-th level.

Definition 2a

As a *partition* $\langle \Phi_k \to \Phi_n \rangle$ we mean set of max-disjoint blocks σP_m what we can denote as follows, $m=n-k+1$

$$\langle \Phi_k \to \Phi_n \rangle = \sigma_1 P_m \ \oplus \ \sigma_2 P_m \ \oplus \ ... \ \oplus \ \sigma_N P_m$$

where N = $\binom{n}{k-1}_F$ members. However blocks $\sigma_1 P_m[\varphi_1, \varphi_2, ..., \varphi_m]$ and $\sigma_2 P_m[\omega_1, \omega_2, ..., \omega_m]$ are max-disjoint iff at least one pair of their sets are disjoint i.e.

$$\exists i \in [m] : \varphi_i \cap \omega_i$$

Definition 2b

We can also consider layer $\langle \Phi_k \to \Phi_n \rangle$ as a sequence of vertices' sets, as follows

$$\langle \Phi_k \to \Phi_n \rangle \equiv \langle \Phi_1, \Phi_2, ..., \Phi_m \rangle$$

where $\Phi_s = [s_F] = \{1, 2, ..., s_F\}$ is a set of vertices on s-th level.

2. Rules to produce partition

These rules are creating a structure similar to a grammar G with set of terminal symbols V_T, non-terminal symbols V_N and relation $P \subseteq (V_N \cup V_T)^+ \times (V_N \cup V_T)^*$ as a set of rules.

Consider any layer $\langle \Phi_k \to \Phi_n \rangle \equiv \langle \Phi_1, \Phi_2, ..., \Phi_m \rangle$, $k, n \in N, k \le n$, $m=n-k+1$.

Definitions

- Set of non-terminal symbols $V_N = \left\{ v \in 2^{\langle \Phi_k \to \Phi_n \rangle} : k \le n \wedge n, k > 0 \right\}$ is a set of layers which we need to change by terminal symbols i.e. sets of certain vertices covered by blocks σP_m
- Set of terminal symbols $V_T = \left\{ u \in 2^{\Phi_s} : s = k, k+1, ..., n \right\}$ is a set of vertices' sets on each of m levels.

Rules of non-terminal symbols

Consider any layer $\langle \Phi_k \to \Phi_n \rangle \equiv \langle \Phi_1, \Phi_2, ..., \Phi_m \rangle$, $v \in V_N$, $u \in V_T$

NI. If $|\Phi_s| = s_F$ for $s=1,2,\ldots,m$ – then layer can partition by one block P_m

$$\langle \Phi_1, \Phi_2, ..., \Phi_m \rangle \to (\varphi_1, \varphi_2, ..., \varphi_m) \qquad : \varphi_s = \Phi_s \text{ for } s=1,2,\ldots,m$$

NII. If m = 1 then our layer (with one level) can be partitioned by $N = |\Phi_1|$ blocks P_1 which cover after 1_F vertices

$$\langle \Phi_1 \rangle \to (u_1) \oplus (u_2) \oplus ... \oplus (u_N) \qquad : |u_s| = 1_F$$

NIII. Else we have the recurrence rule

$$\langle \Phi_1, \Phi_2, ..., \Phi_m \rangle \to \langle \Phi_1, \Phi_2, ..., \Phi_{m-1} \rangle (\varphi_A) \oplus \sigma(\langle \Phi_B, \Phi_1, \Phi_2, ..., \Phi_{m-1} \rangle)$$

where φ_A is a set of m_F first vertices, Φ_B is a set of rest vertices i.e. $\Phi_B = \Phi_m \setminus \varphi_A$, $\Phi_m = \varphi_A \cup \Phi_B \wedge \varphi_A \cap \Phi_B = \emptyset$

Hint 1
The NIII rule is dependent on cobweb-admissible sequence – in this case it's for Natural numbers, because Natural numbers satisfy:

$$n_F = m_F + k_F$$

where $n=m+k$, $m,k \in \mathrm{N} \cup \{0\}$, it is a number of vertices on n-th level. However Fibonacci numbers satisfy:

$$n_F = (m+k)_F = (k+1)_F m_F + (m-1)_F k_F$$

therefore NIII rule for Fibonacci numbers will be separated to $(k+1)_F$ the same terms $\alpha \equiv \langle \Phi_1, \Phi_2, ..., \Phi_{m-1} \rangle (\varphi_A)$ and $(m-1)_F$ the same terms $\beta \equiv \sigma(\langle \Phi_B, \Phi_1, \Phi_2, ..., \Phi_{m-1} \rangle)$:

NIIIF. Recurrence rule

$$\langle \Phi_1, \Phi_2, ..., \Phi_m \rangle \to \underbrace{\alpha \oplus \alpha \oplus ... \oplus \alpha}_{(k+1)_F} \oplus \underbrace{b \oplus b \oplus ... \oplus b}_{(m-1)_F}$$

where φ_A and Φ_B are defined in NIII.

Rules of terminal symbols
These rules is only to modify notation or order of terminal symbols

TI. Concatenate

$$(\varphi_1, \varphi_2, ..., \varphi_m)(\omega_1, \omega_2, ..., \omega_k) = (\varphi_1, \varphi_2, ..., \varphi_m \omega_1, \omega_2, ..., \omega_k)$$

TII. Distributive
Let any $u, v, t \in V_T$

$$(u \oplus v)t = (ut) \oplus (vt)$$

TIII. Order of the members

$$\sigma(\varphi_1, \varphi_2, ..., \varphi_m) = (\varphi_2, \varphi_3, ..., \varphi_m, \varphi_1)$$

3. Example

Consider $\langle \Phi_2 \to \Phi_4 \rangle$, where F – Natural numbers. In notation introduced in this note

$$\langle \Phi_2 \to \Phi_4 \rangle = \langle \Phi_2, \Phi_3, \Phi_4 \rangle = \langle \{1,2\}, \{1,2,3\}, \{1,2,3,4\} \rangle$$

Steeps

$$\langle \{1,2\}, \{1,2,3\}, \{1,2,3,4\} \rangle \to \overbrace{\langle \{1,2\}, \{1,2,3\} \rangle (\{1,2,3\})}^{A} \oplus \overbrace{\sigma(\langle \{4\} \{1,2\}, \{1,2,3\} \rangle)}^{B} \cdots \quad \text{Rule NIII}$$

$$\text{A: } \langle \{1,2\}, \{1,2,3\} \rangle (\{1,2,3\}) \to \left[\overbrace{\langle \{1,2\} \rangle (\{1,2\})}^{C} \oplus \overbrace{\sigma(\langle \{3\}, \{1,2\} \rangle)}^{D} \right] (\{1,2,3\}) \ldots \quad \text{Rule NIII}$$

B: $\sigma(\langle\{4\}\{1,2\},\{1,2,3\}\rangle) \to \sigma(\{4\}\{1,2\},\{1,2,3\}) \to (\{1,2\},\{1,2,3\}\{4\})$. Rule NI + TIII

C: $\langle\{1,2\}\rangle(\{1,2\}) \to ((\{1\}) \oplus (\{2\}))(\{1,2\}) \to (\{1\}\ \{1,2\}) \oplus (\{2\}\ \{1,2\})$. Rule NII + TII

D: $\sigma(\langle\{3\},\{1,2\}\rangle) \to \sigma(\{3\},\{1,2\}) \to (\{1,2\},\{3\})$. Rule NI + TIII

A: $[(\{1\}\ \{1,2\}) \oplus (\{2\}\ \{1,2\}) \oplus (\{1,2\},\{3\})](\{1,2,3\}) \to$

$\to (\{1\}\ \{1,2\}\ \{1,2,3\}) \oplus (\{2\}\ \{1,2\}\ \{1,2,3\}) \oplus (\{1,2\}\ \{3\}\ \{1,2,3\})$. Rule TII

$$\to (\{1\}\ \{1,2\}\ \{1,2,3\}) \oplus (\{2\}\ \{1,2\}\ \{1,2,3\}) \oplus (\{1,2\}\ \{3\}\ \{1,2,3\}) \oplus (\{1,2\},\{1,2,3\}\{4\})$$

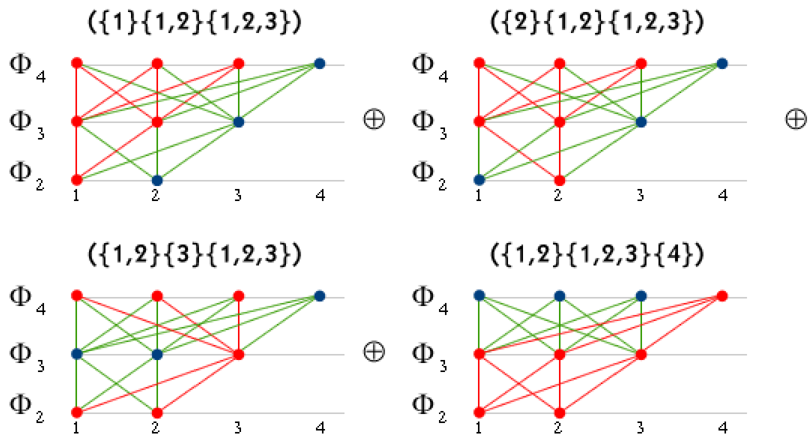

Fig. 1 Display of layer $\langle \Phi_2 \to \Phi_4 \rangle$ as a set of terminal symbols (blocks)

Reference:

[1] A. Krzysztof Kwaśniewski, *On cobweb posets and their combinatorially admissible sequences*, arXiv:math.CO/0512578 v1 26 Dec 2005.

[2] A. Krzysztof Kwaśniewski, *Cobweb posets as noncommutative prefabs* Adv. Stud. Contemp. Math. vol. 14 (1) (2007) 37-47.

[3] M. Dziemiańczuk www.dejaview.cad.pl/cobwebposets.html