Generating Functions' Examples

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1 Introduction

Definition 1.1. *Ordinary* generating function $F(x)$ of $f(n)$ is the formal power series

$$F(x) = \sum_{n\geq 0} f(n)x^n \tag{1.1}$$

while the *exponential* generating function $F(x)$ of $f(n)$ is

$$F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}. \tag{1.2}$$

Let us recall *convolution*

$$\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_n b_{n-k}\right) x^n \tag{1.3}$$

or

$$\left(\sum_{n\geq 0} \frac{a_n x^n}{n!}\right)\left(\sum_{n\geq 0} \frac{b_n x^n}{n!}\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} \binom{n}{k} a_n b_{n-k}\right) \frac{x^n}{n!} \tag{1.4}$$

Definition 1.2. If $F(x) \in \mathbb{C}[[x]]$ satisfies $F(0) = 0$, then we can define for any $\lambda \in \mathbb{C}$ the formal power series

$$(1 + F(x))^\lambda = \sum_{n\geq 0} \binom{\lambda}{n} F(x)^n, \tag{1.5}$$

Example 1.1. Let $\mu(n)$ be the Möbius function of number theory; that is, $\mu(1) = 1$, $\mu(n) = 0$ if n is divisible by the square of an integer greather than one, and $\mu(n) = (-1)^r$ if n is the product of r distinct primes. Find a simple expression for the power series

$$F(x) = \prod_{n\geq 1} (1 - x^n)^{-\mu(n)/n} \tag{1.6}$$

From (1.5) we have that

$$(1 - x^n)^{-\mu(n)/n} = \sum_{k\geq 0} \binom{-\mu(n)/n}{k} (-1)^k x^{kn}$$

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Now apply log to $F(x)$

$$\begin{aligned}\log F(x) &= \log \prod_{n\geq 1} (1-x^n)^{-\mu(n)/n} \\ &= \sum_{n\geq 1} \log(1-x^n)^{-\mu(n)/n} \\ &= \sum_{n\geq 1} \frac{-\mu(n)}{n} \log(1-x^n).\end{aligned}$$

We know that

$$\log(1+x) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} x^k$$

thus

$$\begin{aligned}\log F(x) &= \sum_{n\geq 1} \frac{-\mu(n)}{n} \sum_{k\geq 1} \left(-\frac{x^{kn}}{k}\right) \\ &= \sum_{n\geq 1}\sum_{k\geq 1} \frac{\mu(n)}{kn} x^{kn}\end{aligned}$$

The coefficient of x^m in the above is

$$\frac{1}{m}\sum_{d|m} \mu(d),$$

where the sum is over all positive integers d dividing m. It is well-known that

$$\frac{1}{m}\sum_{d|m} \mu(d) = \begin{cases} 1, & m=1 \\ 0, & \text{otherwise.}\end{cases}$$

Hence $\log F(x) = x$, therefore $F(x) = e^x$ □

Example 1.2. Find the unique sequence $a_0 = 1$, $a_1, a_2 \ldots$ of real numbers satisfying

$$\sum_{k=0}^{n} a_k a_{n-k} = 1 \tag{1.7}$$

Observe that the left siede of the above is a coefficient of convolution of ordinary generating function. Let $F(x) = \sum_{n\geq 0} a_n x^n$, then

$$F(x)^2 = \sum_{n\geq 0} x^n = \frac{1}{1-x}$$

Hence

$$F(x) = (1-x)^{-1/2} = \sum_{n\geq 0} \binom{-1/2}{n} (-x)^n.$$

Therefore the coefficients a_n take a form

$$\begin{aligned}a_n = (-1)^n \binom{-1/2}{n} &= (-1)^n \frac{(-1/2)(-3/2)(-5/2)\cdots(-(2n-1)/2)}{n!} \\ &= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} \qquad \square\end{aligned}$$

Example 1.3. Verify the identity

$$\sum_{i=0}^{n} \binom{a}{i} \binom{b}{n-i} = \binom{a+b}{n}, \tag{1.8}$$

where a, b and n are nonnegative integers.

Observe that the above might be solved with the help of convolution (1.3) of generating function $F(x) = \sum_{i\geq 0} \binom{s}{i} x^i = (1+x)^i$.

$$\begin{aligned}\sum_{k\geq 0}\sum_{i=0}^{n} \binom{a}{i} \binom{b}{n-i} x^k &= \left(\sum_{k\geq 0} \binom{a}{k} x^k\right) \left(\sum_{k\geq 0} \binom{b}{k} x^k\right) \\ &= (1+x)^a(1+x)^b = (1+x)^{a+b} \\ &= \sum_{k\geq 0} \binom{a+b}{k} x^k \qquad \square\end{aligned}$$

2 Binomial posets

Theorem 1. *Let $R(P)$ be Reduced Incidence Algebra over binomial poset P. Then we have $\phi : R(P) \to \mathbb{C}[[x]]$ given by*

$$\phi(f) = \sum_{n\geq 0} f(n) \frac{x^n}{B(n)} \tag{2.1}$$

where $B(n)$ is the total number of maximal chains in n-inverval $[x, y]$ of poset P.

Observation 1. *Let $f(n)$ be the cardinality of an n-interval $[x, y]$ of P. Then*

$$\sum_{n\geq 0} f(n) \frac{x^n}{B(n)} = \left(\sum_{n\geq 0} \frac{x^n}{B(n)}\right)^2 \tag{2.2}$$

Proof. Notice that $\phi(\zeta) = \sum_{n\geq 0} \frac{x^n}{B(n)}$ and $\zeta^2 = \text{card}[x, y]$. $\square$

Observation 2. *If $\mu(n)$ denotes the Möbius function $\mu(x, y)$ for an n-interval $[x, y]$ of P, then we have*

$$\sum_{n\geq 0} \mu(n) \frac{x^n}{B(n)} = \left(\sum_{n\geq 0} \frac{x^n}{B(n)}\right)^{-1} \tag{2.3}$$

Examples:

1. *Ordinary generating function $F(x) = \sum_{n\geq 0} f(n)x^n$*

$$\sum_{n\geq 0} \binom{t}{n} x^n = (1+x)^t. \tag{2.4}$$

2. *Exponential generating function $F(x) = \sum_{n\geq 0} f(n)x^n/n!$*

$$\sum B(n) \frac{x^n}{n!} = \exp\{e^x - 1\}, \qquad \sum D(n) \frac{x^n}{n!} = \frac{e^{-x}}{1-x}. \tag{2.5}$$

Hint. Here $B(n)$ stay for Bell numbers.

3. *Eulerian generating functions* $\sum_{n\geq 0} x^n/n_q!$, where $n_q! = (1+q)\cdots(1+q+\cdots+q^{n-1})$.

$$\sum_{n\geq 0} f(n)\frac{x^n}{n_q!} = \left(\sum_{n\geq 0}\frac{x^n}{n_q!}\right)^2 \tag{2.6}$$

where $f(n)$ - the total number of subspaces of $V_n(q)$, i.e., $f(n) = \sum_k \binom{n}{k}_q$.

4. *Chromatic generating functions* $F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{q^{\binom{n}{2}}n!}$ for $q \in \mathbb{P}$

$$\sum_{n\geq 0} f(n)\frac{x^n}{2^{\binom{n}{2}}n!} = \left(\sum_{n\geq 0}(-1)^n\frac{x^n}{2^{\binom{n}{2}}n!}\right)^{-1} \tag{2.7}$$

where $f(n)$ is the number of acyclic digraphs on n vertices.

References

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